

Equilibrium rotation of semiliquid exoplanets and satellites

Valeri V. Makarov

vvm@usno.navy.mil

US Naval Observatory, 3450 Massachusetts Ave NW, Washington DC 20392-5420, USA

Accepted . Received ; in original form

ABSTRACT

A wide range of exoplanet and exomoon models are characterized by a finite average rigidity and a viscosity much lower than the typical values for terrestrials. Such semiliquid bodies may or may not have rigid crusts with permanent figures. Unlike planets with solid mantles and Earth-like rheology, semiliquid bodies can be captured into stable pseudosynchronous spin resonance, where the average rate of rotation is higher than the synchronous 1:1 resonance. Two basic conditions are derived for capture of planets with a triaxial figure into pseudosynchronous rotation, one related to the characteristic tidal wave number (the product of the tidal frequency by the Maxwell time), and the other to the orbital eccentricity. If a semiliquid object does not satisfy either of the two conditions, it is captured into the synchronous resonance. For nearly axially symmetric bodies, only the first condition is in place, and the other is much relaxed, so they should predominantly be pseudosynchronous. It is also pointed out that the equilibrium pseudosynchronous rotation rate can not reach the widely used asymptotic value from the constant time lag model but is in reality closer to the synchronous spin.

1. Introduction

Most of the Solar system bodies subject to significant tidal interaction are captured into the 1:1 spin-orbit resonance, also known as synchronous rotation, when the body's average sidereal spin rate is exactly equal to the orbital motion. As a result of this resonance, the body is permanently facing the perturber with the same side, a phenomenon well known to us from the Moon. A notable exception to this rule is Mercury, which is captured in a stable 3:2 resonance, which makes one Mercurian solar day as long as two Mercurian solar years. As was shown by Hut (1980), synchronous rotation and a circular orbit is the terminal point of the tidal spin-orbit evolution; however, higher stable spin-orbit resonances are possible and even likely for super-Earth exoplanets of presumably terrestrial composition (Makarov & Berghea 2014). Like Mercury, such planets, once caught in a super-synchronous resonance, should stay there indefinitely long, unless an external impulse, such as a giant impact, drives the planet out of this locally stable state. Most of the Solar system moons, on the other hand, are locked

in the 1:1 resonance, and no credible evidence was found of an equilibrium rotation with a non-resonant rate above the synchronous rate.

Investigation of equilibrium near-synchronous rotation states was pioneered by Goldreich (1966) for the *ad hoc* constant- Q tidal model, in which the tidal torque is independent in magnitude of the perturbation frequency. For a long time the work-horse of the tidal dynamics, the Constant Time Lag (CTL) model of tidal dissipation assumes that the secular tidal torque acting on the perturbed body can be approximated by a linear dependence on tidal frequency ν with a negative slope (analogous to a regular force of friction), but shifted upward for a non-vanishing eccentricity (Murray & Dermott 1999). As a result of the shift, the secular tidal torque is positive (accelerating in the prograde sense) at the point of resonance, i.e., where the spin rate exactly equals the mean orbital motion, $\dot{\theta} = n$. The secular term, which acts as friction in the balance of forces, vanishes at (Hut 1981; Rodríguez et al. 2008; Ferraz-Mello et al. 2008; Williams & Efroimsky 2012)

$$\dot{\theta} \simeq (1 + 6e^2)n \quad (1)$$

If the CTL model were applicable, in the absence of other forces, the moons of the Solar system would have been involved in a slow prograde rotation relative to their host planets. The commonly accepted explanation why this does not happen¹ refers to the presence of a geometrical triaxial torque caused by the elongation of the permanent figure. This torque emerges whenever the axis of the smallest inertia moment begins to deviate from the instantaneous direction to the perturber; thus, it acts as a restorative force directed toward the point of exact synchronism.

Pseudo-synchronous equilibrium emerges in other, more advanced and complex tidal models as well. Ferraz-Mello (2013) considered a rheological model dominated by Newtonian creep mechanism of dissipation and found an equilibrium tide at a super-synchronous rate of rotation, although the shift turned out to be frequency-dependent in that case, as opposed to simply $6e^2n$. The hybrid viscoelastic model with self-gravitation and Andrade creep developed by Efroimsky (2012a,b), also permits super-synchronous equilibria in principle. However, this state was shown to be practically impossible for bodies of terrestrial composition, characterized by large effective viscosities and long Maxwell times (Makarov & Efroimsky 2013), see also (Correia et al. 2014). On the other hand, a wide range of plausible rheological models for exoplanets and exomoons, as well as for smaller satellites of the Solar system, can be best represented by the viscoelastic paradigm with low values of viscosity but finite rigidity. The first important class of such objects is a terrestrial body that has been heated up by strong tidal forces past the point of solidus in the deeper layers of the mantle.

¹There are lingering doubts about the exact synchronism of Europa though (Geissler et al. 1998)

2. Torques

In the following, we consider tides and forces acting on one of the components in a system of two orbiting bodies. The component under consideration is called in this paper “perturbed body”, or “rotating body”, and in practice can be a planet or a satellite. The other component is called “perturbing body”, or simply “perturber”, and it can be a star or a planet. The possible tides or deformations of the perturber are not involved in the analysis; it is effectively a point-mass with only a central gravitational action on the perturbed body.

Comprehensive equations for the polar tidal torque (i.e., the torque component directed along the axis of rotation), including the fast oscillating terms, can be found, for example, in (Makarov & Berghea 2014). Here we only summarize the main equations necessary for the following analysis of equilibrium rotation. The non-polar component of perturbation are ignored in this analysis, which is equivalent to setting the orbit obliquity on the perturbed body’s equator to zero. The instantaneous torque acting on a rotating body is the sum of the triaxial torque caused by the quadrupole inertial momentum, and the tidal torque caused by the dynamic deformation of its body. Planets and moons of terrestrial composition have a permanent figure, which results in the three principal moments of inertia A , B , and C (the C denoting traditionally the largest moment of inertia) being unequal to one another. The degree of elongation of the permanent figure, or the departure from the dynamical axial symmetry, can be represented by the parameter $(B - A)/C$, which ranges from ≈ 0.1 for smaller moons to $\approx 10^{-5}$ for the larger rocky planets. The angular acceleration is related to the polar components of these torques by

$$\ddot{\theta} = \frac{T_{\text{TRI}} + T_{\text{TIDE}}}{\xi M_2 R^2} \quad (2)$$

with θ being the rotation angle of the perturbed body reckoned from the axis of its largest elongation, corresponding to the moment A , M_2 is its mass, and ξ being the coefficient of inertia, $C/(M_2 R^2)$. Possible variations of the moments of inertia on longer time scales are not considered in this paper. The equations of the triaxial torque T_{TRI} have been exploited in the literature on many occasions (Danby 1962; Goldreich & Peale 1968; Murray & Dermott 1999). Neglecting the precession and nutation of the planet, the angular acceleration due to the triaxial torque is

$$\ddot{\theta}_{\text{TRI}} = -\frac{3}{2} \frac{B - A}{C} \frac{a^3}{r^3} n^2 \sin 2(\theta - f), \quad (3)$$

where r is the instantaneous distance of the perturbed body from the perturber, a is the semi-major axis, n is the orbital mean motion, and f is the instantaneous true anomaly. This formula can be more conveniently rewritten as an infinite series of mean anomaly (\mathcal{M}) harmonics:

$$\ddot{\theta}_{\text{TRI}} = -\frac{3}{2} \frac{B - A}{C} n^2 \sum_j G_{20(j-2)} \sin(2\theta - j\mathcal{M}), \quad (4)$$

where G_{lpq} are Kaula’s functions of eccentricity related to Hansen’s coefficients through $G_{20(j-2)} = X_j^{-3,2}$. Note that the summation in Eq. 4 is over $j = -\infty, \dots, +\infty$, and $G_{20(-2)} = 0$.

A simplified equation for the angular acceleration caused by tidal torque can be written as (Efroimsky 2012a, Eqs. 106-108)

$$\ddot{\theta}_{\text{TIDE}} = \frac{3}{2} \frac{M_1}{\xi M_2} n^2 \left(\frac{R}{a} \right)^3 \sum_{q=-1}^4 G_{20q}(e) \sum_{j=-1}^4 G_{20j}(e) [K_c(2, \nu_{220q}) \text{Sign}(\omega_{220q}) \cos((q-j)\mathcal{M}) - K_s(2, \nu_{220q}) \sin((q-j)\mathcal{M})], \quad (5)$$

assuming $M_2 \ll M_1$ where M_1 is the mass of the perturbing body, and

$$\begin{aligned} K_c(2, \nu_{220q}) &\equiv k_2(\nu_{220q}) \sin \epsilon_2(\nu_{220q}) \quad , \\ K_s(2, \nu_{220q}) &\equiv k_2(\nu_{220q}) \cos \epsilon_2(\nu_{220q}) \quad , \end{aligned} \quad (6)$$

k_2 being the degree-2 dynamical Love number, $\epsilon_2(\nu_{220q})$ the positive definite, mode-dependent tidal phase lag and ν_{220q} being the positive definite physical forcing frequency related to the semidiurnal Fourier mode through

$$\nu_{220q} = |\omega_{220q}| = |(2+q)n - 2\dot{\theta}|. \quad (7)$$

One of the functions, K_c , has been scrutinized in the literature (Efroimsky 2012a,b). Historically, it is denoted as k_2/Q , also dubbed *kvalitet* elsewhere, the Q value being related to the amount of energy dissipated over one cycle of rotation. Secular (i.e., non-averaging over one orbital period) acceleration emerges from the K_c term when $q = j$.

Within the simplifying assumption of a spherical, incompressible, and homogeneous body, and a purely analytical approach adopted in this paper, the corresponding *kvalitet* can be written as (Efroimsky & Lainey 2007; Efroimsky 2012a,b)

$$K_c(2, \nu) = -\frac{3}{2} \frac{\mathcal{A}_2 \nu \tau_M \Im(\nu)}{(\Re(\nu) + \mathcal{A}_2 \nu \tau_M)^2 + \Im^2(\nu)} \quad (8)$$

where

$$\mathcal{A}_2 = 4\pi \frac{19}{6} \frac{R^4 \mu}{\mathcal{G} M_2^2}, \quad (9)$$

\mathcal{G} being the gravitational constant, and

$$\begin{aligned} \Re(\nu) &= \nu \tau_M + \nu^{1-\alpha} \tau_M \tau_A^{-\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(1+\alpha) \\ \Im(\nu) &= -1 - \nu^{1-\alpha} \tau_M \tau_A^{-\alpha} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma(1+\alpha) \end{aligned} \quad (10)$$

The $\Re(\nu)$ and $\Im(\nu)$ dimensionless functions are the real and imaginary parts of the normalized dimensionless complex compliance, respectively, multiplied by tidal wave number $\nu \tau_M$ to avoid numerical singularity at zero frequency. In these equations, R is the radius of the perturbed body, M_2 is its mass, τ_M is the Maxwell time, τ_A is the Andrade time, μ is the unrelaxed shear modulus. The frequency ν is used hereafter as a shorthand version of ν_{220q} , so the functions remain dependent on both the frequency and the tidal mode. The dimensionless Andrade

parameter α takes values from ~ 0.16 for a partial melt to ~ 0.4 for cold minerals. The parameter \mathcal{A} , sometimes called effective rigidity in the literature, is related to the definition of the static Love number (Goldreich & Peale 1968). The second terms of the complex compliance components represent the contribution of the Andrade creep mechanism.

These basic equations greatly simplify for the case of semiliquid bodies considered in this paper. Recalling that the Maxwell time is defined as $\tau_M = \eta/\mu$, where η is the effective viscosity, and that the Andrade creep is perhaps insignificant for such bodies, one can use the constraints $\tau_A = \infty$ and $\tau_M n \ll 1$. These simplify the secular kvalitet to

$$K_c(2, \nu) \equiv k_2(\nu) \sin \epsilon_2(\nu) = \frac{3}{2} \frac{\mathcal{A}_2 \lambda}{\lambda^2 (1 + \mathcal{A}_2)^2 + 1}, \quad (11)$$

where $\lambda = \tau_M \nu$ is a dimensionless quantity hereafter termed the tidal wave number. This function is strongly dependent on tidal frequency in narrow vicinities of spin-orbit resonances $(2 + q)n = 2\dot{\theta}$ for integer q . The characteristic kink-shape of the near-resonant torque has the largest amplitude in the 1:1 resonance for small and moderate orbital eccentricities. A steep decline between the two peaks (Fig. 1) occupies a narrow band of frequencies for realistic rheologies. Despite the relatively small amplitude of the kink (compared to a typical amplitude of the triaxiality-caused torque), it acts as an efficient trap for a planet trying to traverse the resonance. The maximum kvalitet is reached at

$$\lambda_{\max} = \frac{1}{1 + \mathcal{A}_2}, \quad (12)$$

where its value is

$$K_c(2, \nu_{\max}) = \frac{3}{4} \frac{\mathcal{A}_2}{1 + \mathcal{A}_2}. \quad (13)$$

3. First condition of pseudosynchronous equilibrium

As is evident from Eq. 12, the locale of maximum tidal force in frequency is inversely proportional to the effective Maxwell time. Relatively cold terrestrial mantles are characterized by large values of viscosity (e.g., Doremus 2002), and, thus, long Maxwell times. For example, the tidal response of Mercury was modeled with a $\tau_M = 500$ years (Noyelles et al. 2014), and the suggested value for the Moon is ~ 10 years as estimated from the observed tilt of the mean principal axis (Makarov 2013). Icy satellites, on the other hand, can have much shorter Maxwell times owing to the relatively low viscosity of pressurized ice ($\sim 10^{15}$ Pa·s). The slope of the central part of the secular torque between the opposing peaks (Fig. 1, left), which can be roughly approximated with an inclined line, is therefore proportional to the Maxwell time. Within this segment of the kink-shape, the secular torque acts as a quasi-linear friction force on any deviations of the spin rate from the equilibrium value. The point of equilibrium, or zero secular torque, is associated with the root of the torque curve, which is offset from

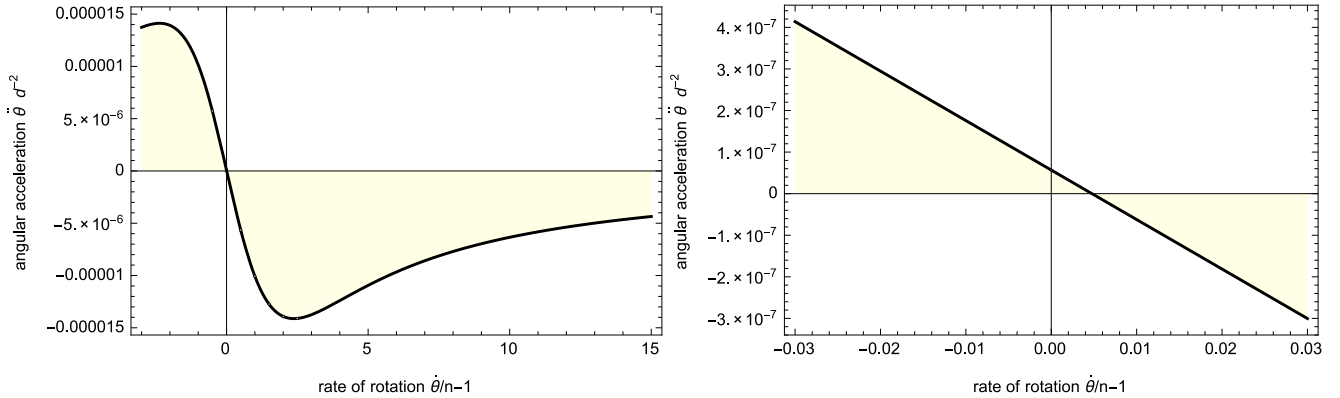


Fig. 1.— Angular acceleration of Titan caused by the secular component of tidal torque (Eq. 5) with the assumed parameters listed in Table 1 and with a Maxwell time of 0.1 days. Left: a large-scale view of the dependence showing the main “kink” corresponding to the 1:1 resonance term, ω_{2200} . Right: the same dependence on much smaller scale to demonstrate the offset of null torque from the exact 1:1 resonance. The kinks associated with the other resonances $\dot{\theta}/n = 0.5, 1.5, 2.0, 2.5, \dots$ are too small in amplitude and smoothed out to be discerned.

the point of synchronous resonance because of the positive, nearly constant, component due to the second-largest term $q = 1$ for a finite eccentricity (Fig. 1, right). The tidal torque drives the rotating body away from the point of synchronism $\nu_{2200} = 0$ toward the point of equilibrium $\nu_{2200} = \nu_{\text{pseudo}}$ when the frequency takes values within the narrow segment between these points. Obviously, the point of pseudosynchronous equilibrium, where the secular torque vanishes, can not be outside of the linear segment bracketed by the opposite peaks. Therefore, the asymptotic value of $(1 + 6e^2)n$ can only be realized when this frequency is much smaller than the location of the peak $\tau_M^{-1}(1 + \mathcal{A}_2)^{-1}$, i.e., in liquid or semiliquid bodies. For inviscid planets with Maxwell time comparable to, or greater than, the orbital period of the perturber, the point of equilibrium collapses to very small values. This practically eliminates the feasibility of pseudosynchronous equilibrium, as any small perturbation, including physical libration, will drive the body into the globally stable synchronous rotation.

Fig. 2 shows the relative, normalized frequency of pseudosynchronous equilibrium as a function of tidal wave number for the Titan model (Table 1). This graph is obtained by computing the roots of Eq. 5 for the secular part of tidal acceleration ($q = j$) and for different values of τ_M . The previously known point of $(1 + 6e^2)n$ from the CTL formalism, indicated by the horizontal line in the graph, is only an asymptotic limit achieved at $\lambda \rightarrow 0$. Realistic viscoelastic bodies with self-gravitation attain the pseudosynchronous equilibrium at any rotation rate between n and $(1 + 6e^2)n$ depending on the effective wave number $\tau_M n$.

As is shown in Fig. 2, the point of equilibrium becomes close to the CTL limit at wave numbers as small as 0.1. This implies a Maxwell time of 10 days or shorter for the Titan model (Table 1). The fact that Titan is in synchronous rather than pseudosynchronous rotation can

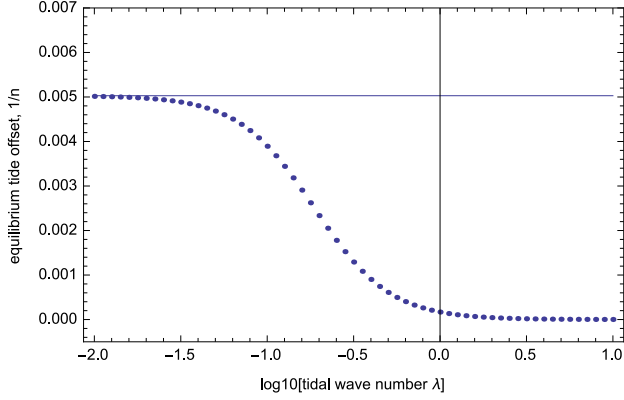


Fig. 2.— Normalized frequency of equilibrium, i.e. of vanishing secular tidal torque, as a function of tidal wave number λ . The asymptotic value $6e^2$ is shown with the horizontal line.

Table 1: Parameters of the tidal model for Titan as a tentative semiliquid satellite.

Name	Description	Units	Values
ξ moment of inertia coefficient		0.34
R radius of planet	m	$2.576 \cdot 10^6$
M_2	mass of the perturbed body (satellite)	kg	$1.3452 \cdot 10^{23}$
M_1	mass of the perturbing body (planet)	kg	$5.68 \cdot 10^{26}$
a semimajor axis	m	$1.2219 \cdot 10^9$
n mean motion, i.e. $2\pi/P_{\text{orb}}$	d^{-1}	0.394
e orbital eccentricity		0.0288
$(B - A)/C$ triaxiality		$1.3 \cdot 10^{-4}$
\mathcal{G} gravitational constant	$\text{m}^3 \text{ kg}^{-1} \text{ yr}^{-2}$	66468
τ_M Maxwell time	yr	var.
τ_A Andrade time	yr	∞
μ unrelaxed rigidity modulus	Pa	$3 \cdot 10^9$
α the Andrade parameter		0.2

be interpreted as Titan’s effective Maxwell time being longer than that value. However, other circumstances of purely dynamical, rather than rheological, character can prevent the capture of inviscid bodies into pseudosynchronism.

4. Second condition of pseudosynchronous equilibrium

Capture into spin-orbit resonances, including the 1:1 (synchronous) resonance, is a probabilistic process, conditioned by the balance of restorative and damping forces similar to a nonlinear, driven rotating pendulum with friction (Goldreich & Peale 1966, 1968). The subtle conditions of a *passage* through spin-orbit resonances were numerically explored by (Makarov 2012). In general, at each resonance $\dot{\theta}/n = 1:2, 1:1, 3:2, 2:1, \dots$, the resonant term of secular torque will be biased (or offset from zero) by the contribution of the other secular ($q = j$) terms. For example, the bias at the 1:1 resonance in secular torque is positive (see Fig. 1, right), mostly defined by the left-hand shoulder of the 3:2 secular term, for moderate eccentricities. Physically, the positive bias is generated by the fact that the perturber moves with a higher than n angular velocity during one half of the orbital period, and the bodies are closer to each other during that part of the orbit than the other half. The emerging positive, or accelerating in the prograde sense torque, tends to spin up the synchronized perturbed body (Murray & Dermott 1999). On the contrary, the bias at the 3:2 resonance is usually negative (decelerating) at moderate eccentricities, mostly defined by the right-hand shoulder of the 1:1 term. It is in fact this bias that generates the possibility of non-resonant equilibrium, or pseudosynchronism. The bias can well be greater in magnitude than the half-amplitude of a resonant kink, in which case there is no equilibrium of secular torque at all (however, capture into that resonance may still be possible). Obviously, the bias acts as a driving force, trying to drag the rotating body through a resonance, whereas the near-linear inclined segment between the peaks acts as a damping force, or friction, working to get rid of the excess kinetic energy and to capture the body into the resonance.

Returning to the useful analogy with a driven rotating pendulum, the mechanics of a resonance encounter can be visualized as a gradual slowing of rotation due to the secular tidal torque. With each cycle, the minimum angular velocity at the top of the circular path becomes smaller, until the pendulum stops close to the top point, no longer able to reach it. It reverses the motion, suddenly switching from circulation to oscillation. The first full-amplitude swing in the counter direction determines the fate of the pendulum. If the work done by the friction force is greater than the work done by the driving torque, the pendulum will not be able to come over the top in the opposite direction either, and it will continue to oscillate around the point of equilibrium with an ever decreasing amplitude due to the damping. In the numerical simulation shown in Fig. 3 (left), this capture into the synchronous resonance takes place at about 137000 days. On the other hand, when the driving torque is strong relative to the friction, the pendulum will pass over the top and continue to circulate in the opposite direction.

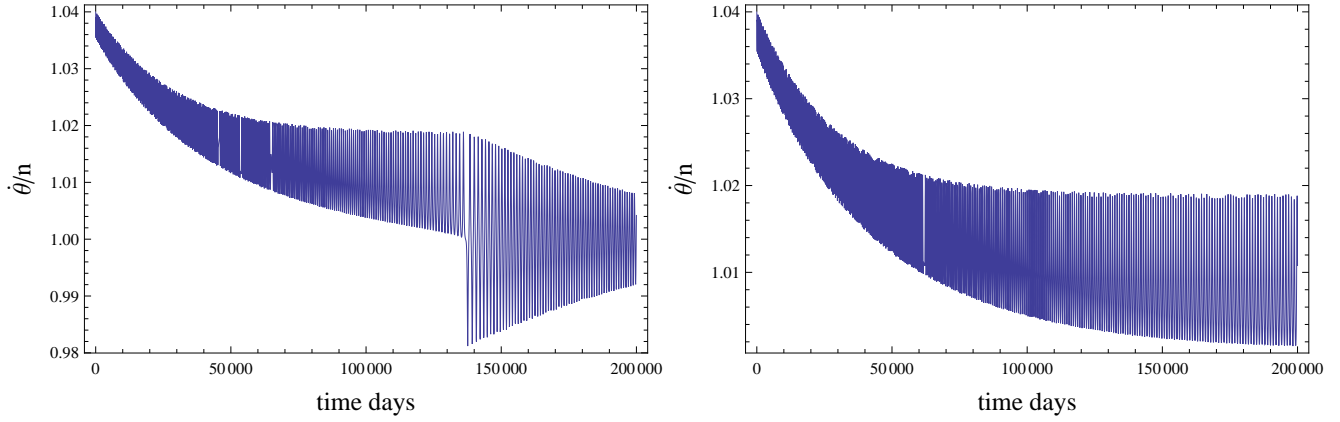


Fig. 3.— Numerical integration of Titan’s spin rate approaching the 1:1 spin-orbit resonance for two different values of orbital eccentricity. Left: capture into synchronous rotation with $e = 0.045$. Right: capture into pseudosynchronous rotation with $e = 0.046$.

Capture into a pseudosynchronous equilibrium state may occur when the secular tidal torque averaged over one circulation period, vanishes due to the offset in relative frequency $\dot{\theta}/n - 1$ (Fig. 1, right) while the minimum velocity at the top of the trajectory is still greater than the resonant value. In other words, the pendulum still rotates in the prograde direction but the secular torque is balanced out. This situation is depicted in Fig. 3 (right) for the same set of model parameters. The mean velocity $\dot{\theta}$ remains above $1\,n$ indefinitely long, because the lowest velocity over a free libration cycle can not reach the resonant value. Should the resonant value be reached, an immediate transition into the synchronous state happens.

The emerging condition of pseudosynchronous equilibrium can be quantified using the pre-resonant trajectory for free librations. Let us introduce, following (Goldreich & Peale 1968),

$$\gamma = 2\theta - 2\mathcal{M}, \quad (14)$$

which is the sidereal libration angle multiplied by 2. Large-amplitude free librations of bodies with a permanent figure are driven by the triaxial torque (Eq. 3). For the case of small eccentricity, which is valid for the majority of close exoplanets detected so far, as well as for the majority of Solar system satellites, only the dominating term from Eq. 4 can be retained leading to

$$\ddot{\gamma} \simeq -3 \frac{B-A}{C} n^2 G_{200}(e) \sin \gamma. \quad (15)$$

Note that this approximation can only be used for $e < 0.1$. From the consideration of energy conservation

$$\dot{\gamma}^2 = \dot{\gamma}_0^2 + 2\epsilon^2 n^2 (1 + \cos \gamma), \quad (16)$$

where $\epsilon = \sqrt{3G_{200}(e)(B-A)/C}$, and $\dot{\gamma}_0$ is the minimum libration velocity at $\gamma = \pi$. Equilibrium is achieved when the kinetic energy averaged over one libration period is constant, which

requires the work done by the secular tidal torque to be zero, i.e.,

$$\int_0^{2\pi} \ddot{\theta}_{\text{TIDE}} d\gamma = 0. \quad (17)$$

Per condition of pseudosynchronism number one (Section 3), the tidal wave number of the resonant mode should be quite small, in which case the kvalitet in Eq. 11 becomes a linear function of ν

$$K_c(2, \nu) \simeq \frac{3}{2} \mathcal{A}_2 \lambda, \quad (18)$$

and the secular tidal acceleration can be approximated quite well by a linear function,

$$\ddot{\theta}_{\text{TIDE}} \propto -(\dot{\gamma} - \dot{\gamma}_{\text{pseudo}}). \quad (19)$$

Combining Eqs. 16, 17, and 19 obtains

$$\begin{aligned} \dot{\gamma}_{\text{pseudo}} &= \frac{1}{2\pi} \int_0^{2\pi} (\dot{\gamma}_0^2 + 2\epsilon^2 n^2 (1 + \cos \gamma))^{1/2} d\gamma \\ &= \frac{2}{\pi} \left[\sqrt{\dot{\gamma}_0^2 + 4\epsilon^2 n^2} \mathcal{E} \left(\frac{4\epsilon^2 n^2}{\dot{\gamma}_0^2 + 4\epsilon^2 n^2} \right) \right] \end{aligned} \quad (20)$$

where \mathcal{E} stands for the complete elliptic integral. Because of the character of the secular tidal torque (Fig. 1, right), the planet will decelerate if the relative root-mean-square (RMS) velocity over one libration period is higher than $\dot{\gamma}_{\text{pseudo}}$. By the same argument, the planet will accelerate if its RMS velocity is below the point of pseudosynchronism, unless it is captured into the 1:1 resonance. To avoid the resonance, the minimum velocity should be positive, $\dot{\gamma}_0 > 0$. In the marginal case $\dot{\gamma}_0 = 0$, the period of free libration becomes infinitely long, because the pendulum stops in the unstable equilibrium $\ddot{\gamma} = \dot{\gamma} = 0$ and $\gamma = \pi$. The time-average spin rate, therefore, nullifies. Remarkably, the time-average kinetic energy or the RMS velocity do not nullify, in that as the minimum velocity approaches 0, $\dot{\gamma}_0 \rightarrow 0$, the RMS velocity approaches a finite positive number, namely,

$$\dot{\gamma}_{\text{pseudo}} \rightarrow \frac{4}{\pi} \epsilon n. \quad (21)$$

Introduce a nominal eccentricity e_{crit} such that the secular component of tidal torque vanishes at a frequency $6e_{\text{crit}}^2 n$. As discussed in Section 3, $0 < e_{\text{crit}} < e$. By this definition, the relative spin rate $\dot{\theta}_{\text{pseudo}} - n$ at pseudosynchronism should be equal to $6e_{\text{crit}}^2 n$, where $e_{\text{crit}} \rightarrow e$ asymptotically for $\lambda \rightarrow 0$. Taking into account Eq. 14, we finally arrive at the second necessary condition of pseudosynchronous equilibrium:

$$e > \left(\frac{\sqrt{B-A}}{\sqrt{3C}\pi} \right)^{1/2}. \quad (22)$$

A similar condition, only different by a constant factor $\sqrt{19}/2$, was derived by Goldreich (1966) for the rough constant- Q tidal model.

5. Calculations for a model of Titan

Titan is one of the largest satellites in the Solar system, and a good representation of the class of icy moons (Husmann et al. 2010). Objects like Titan can be common in exoplanet systems at relatively larger separations from their hosts, bridging the gap between ice-dominated and terrestrial (silicate) objects. A series of numerical integrations was performed for this study within the uniform body approximation for a set of parameters listed in Table 1. The degree of triaxiality $(B - A)/C$ was deduced from the numbers provided by Bills & Nimmo (2008). The Andrade time was set equal to infinity, thus nullifying the effects of inelastic creep of the Andrade type. For most simulations, the Maxwell time was assumed to be a rather short 0.1 days, which, for an effective shear modulus of $3 \cdot 10^9$ Pa (suggested by F. Nimmo, priv. comm. 2013), implies a rather low effective viscosity of the dissipating layer of $2.6 \cdot 10^{13}$ Pa.s. This may still be a plausible value for a loosely bound mixture of rock and ice, as well as partially molten mantles. A short Maxwell time is required to realize the pseudosynchronous equilibrium according to the first condition (Section 3), i.e., $\tau_M \nu \lesssim 0.1$. The asymptotic RMS velocity of a pseudosynchronously rotating body is $\dot{\theta}_{\text{pseudo}} = (1 + 6e^2)n$, hence the corresponding tidal frequency in the main mode is $\nu = 12e^2n$, which amounts to 0.004 d^{-1} for the present day eccentricity, and to 0.009 d^{-1} for e_{crit} (Section 4). Therefore, for the Titan model in question, the first condition of pseudosynchronism is fulfilled for Maxwell times shorter or approximately equal to 10 days. We note that the first condition can be re-written for the effective viscosity to be

$$\eta \lesssim \frac{0.1 \mu}{12 e^2 n}. \quad (23)$$

Interestingly, for a fixed finite viscosity, the eccentricity can not be very large lest the pseudosynchronism equilibrium collapses by the first condition. Therefore, there is a certain interval of effective viscosity for pseudosynchronism to work. This interval vanishes at a characteristic viscosity

$$\eta_{\text{crit}} = \frac{0.1 \sqrt{3} \pi \mu}{12 \sqrt{(B - A)/C} n} \approx 0.045 \frac{\mu}{\sqrt{(B - A)/C} n}, \quad (24)$$

so that for larger values of viscosity, this type of equilibrium is practically impossible. For our Titan model, the critical viscosity turns out to be $2.6 \cdot 10^{15}$ Pa.s. This is much smaller than the typical effective viscosity of cold terrestrial mantles ($\eta_{\text{Earth}} \approx 1.3 \cdot 10^{21}$ Pa.s). On the other hand, due to the Arrhenius-type dependence of viscosity on temperature (e.g., Doremus 2002), the average viscosity drops to $\simeq 1.3 \cdot 10^{15}$ Pa.s at the bottom of the Earth's mantle (temperature 4000 K, pressure 136 GPa) close to the liquidus, according to both the models presented by Shoji & Kurita (2014) and Stamenković et al. (2012). Similarly small pre-solidus viscosity is reached in the models considered by Henning & Hurford (2014). With a fixed μ , planets and satellites with smaller triaxial deformation or more distant from their hosts have more room for a locally stable non-synchronous state, in accordance with Eq. 24. The triaxial deformation of smaller Saturnian satellites is so large (Tiscareno et al. 2009), that it probably eliminates the possibility of pseudosynchronism.

Several numerical simulations of spin evolution for the Titan model were performed to

verify the second condition of pseudosynchronism (Section 4). The input parameters were kept as in Table 1 except for the eccentricity, which varied between 0.02 and 0.06. Each integration solves numerically an ordinary differential equation of second order (Eq. 2) for the angular acceleration including both the triaxial (Eq. 3) and tidal (Eq. 5) torques. The initial conditions were set to $\theta(0) = 0$, $\dot{\theta}(0) = 1.4n$, so that the moon would start far enough from and above the 1:1 resonance to allow for a free, dissipative descent into an equilibrium. The standard Runge-Kutta integration algorithm in Mathematica was employed with an integration step of 0.1 days and an accuracy goal of 9 significant digits. The tidal torque included both the secular and oscillating terms (although the latter are probably negligible in the presence of much larger free librations), and the summation of tidal modes was performed over indices $j = -1, 0, \dots, 4$, as per Eq. 5, which is quite sufficient for such small eccentricities. The length of integration was set at 200000 days, which would assure a capture into one or the other equilibrium.

Fig. 3 shows the resulting relative spin rate for two fixed values of eccentricity, $e = 0.045$ (left) and $e = 0.046$ (right). This small change in the orbital eccentricity results in a dramatically different outcome of the encounter with the dominating resonance. In the former case, the spin rate at the minimum of a free libration cycle reaches the resonance $1n$, followed by immediate capture into that resonance. In the latter case, the point of equilibrium torque rotation is far enough from $1n$, the minimum spin rate can never quite reach the resonance, and the body remains in a super-synchronous rotation. According to Eq. 22, the theoretical threshold eccentricity enabling pseudosynchronism is 0.0449. These computations show that the actual threshold eccentricity is between 0.045 and 0.046. The cause for this slight difference is in the asymptotic character of the assumed equilibrium velocity $(1 + 6e^2)n$ (Fig. 2).

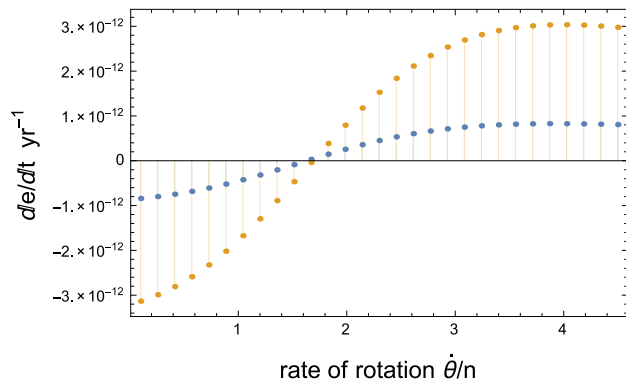


Fig. 4.— Rate of secular eccentricity change for the Titan model (Table 1) with two fixed values of eccentricity: $e = 0.1$ (outer curve) and $e = 0.0288$, or the current value (inner curve).

6. Escapes from equilibrium

Our considerations have been thus far decoupled from the orbital evolution processes caused by the secular components of the polar tidal torque. A comprehensive analysis of the problem should, in principle, include the total energy and angular momentum of the system compounded of the rotational and orbital motion of the bodies. The classic Kaula’s approach, on the other hand, is based on the non-inertial reference frame related to the equator of the perturbed body, and is not amenable to a reformulation in a barycentric non-rotating reference frame. Some justification to this approach is provided by the long-known fact that the characteristic time of spin rate evolution is typically much shorter than the time of orbit evolution driven by tides (Zahn 1977; Auclair-Desrotour et al. 2014). On the time scales of the order of 10^5 years (Fig. 3), it is completely legitimate to ignore the secular changes in the orbital parameters. On the time-scales of a stellar age, however, these changes should be taken into account.

The orbital eccentricity, in particular, is a crucial parameter involved in the conditions of pseudosynchronism. Using Kaula’s frequency decomposition of the perturbing potential approach (Kaula 1964; Lambeck 1975; Remus et al. 2012), we have for the rate of eccentricity from the dominating $l = 2$ terms²

$$\begin{aligned} \left\langle \frac{de}{dt} \right\rangle &= -\frac{n\sqrt{1-e^2}}{(M_1+M_2)a^5 e} \sum_{m=0}^2 \frac{(2-m)!}{(2+m)!} (2 - \delta_{0m}) \\ &\times \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} G_{2pq}^2(e) [\sqrt{1-e^2}(2-2p+q) - (2-2p)] \\ &\times \left[M_2 R_1^5 F_{2mp}^2(i_1) K_1(2, \nu_{2mpq}^{(1)}) \text{Sign}(\omega_{2mpq}^{(1)}) + M_1 R_2^5 F_{2mp}^2(i_2) K_2(2, \nu_{2mpq}^{(2)}) \text{Sign}(\omega_{2mpq}^{(2)}) \right] \end{aligned} \quad (25)$$

where K_1 and K_2 are the tidal quality functions from Eq. 8 for the primary and the secondary components (Saturn and Titan, in our case), respectively, i_1 and i_2 are the obliquities of the orbital plane on the respective equators, and R_1 and R_2 are the radii. Ignoring the term corresponding to the dissipation in Saturn, which is probably small³, this equation implies a negative rate of eccentricity everywhere in the neighborhood of the 1:1 resonance for a range of past eccentricities, see Fig. 4. Thus, a Titan in pseudosynchronous or synchronous equilibrium should be slowly damping the eccentricity, barring third-body interactions. The upper bound equilibrium frequency, being proportional to e^2 , will decline faster, squeezing the range of pseudosynchronism. Eventually, the critical eccentricity is achieved and the body falls into the 1:1 resonance.

This process was confirmed by numerical simulations. Fig. 5 depicts the results of a numerical simulation similar to the ones in Fig. 3 but for a longer time, and with a variable eccentricity. The initial parameters were set to $\theta(0) = 0$, $\dot{\theta}(0) = 1.03 n$, while a slowly declining

²For comparison, see similar equation derived for the CTL model in (Mignard 1980, 1981).

³See, however, a small Q for Saturn in (Lainey et al. 2012) estimated from astrometric observations of its satellites.

eccentricity was modeled as $0.06 \times \cos(3 \cdot 10^{-6} t)$, bringing it down to approximately 0.04 by the end of the integration. If the initial eccentricity were kept fixed during the integration span, the model body would descent into the pseudosynchronous state and remain there indefinitely long staying clear of the resonance. But the declining eccentricity pushed the RMS velocity down, the minimum free libration velocity reached $1n$, and immediate capture into the 1:1 resonance followed.

7. Rotationally symmetric planets

Thus far, we have considered celestial bodies which have nonzero elongation in the distribution of mass and in shape, when in the sequence of principal moments of inertia $A \leq B \leq C$, the smallest two are unequal, $B > A$. This class of objects is of special practical interest, as it presumably includes most Earth-sized planets and smaller satellites. Only the greater planets of the Solar system do not fall into this category. To a first-order approximation, moons with deep oceans covered by a rigid surface crust can also be described by our model, as long as the crust is not perfectly symmetric and is intact as a whole.⁴ Other plausible configurations include planets with massive surface oceans but rigid cores, the latter likely to have nonzero triaxial deformation. Finally, close-in terrestrial planets can be molten to the surface by tidal dissipation or stellar irradiation, in which case they can not maintain a permanent figure. A bulge can only be raised by external perturbations, and should be dynamic and transient in nature. It is not obvious if the viscoelastic model in question is at all valid for such bodies. However, the commonly employed CTL model is mathematically similar to the Maxwell model in the linear regime. Therefore, the case of $A = B$ may at least be of theoretical interest,

⁴A more accurate and consistent treatment of bodies with subsurface oceans requires more complex modeling including tidal wave propagation and boundary friction, best handled by numerical methods.

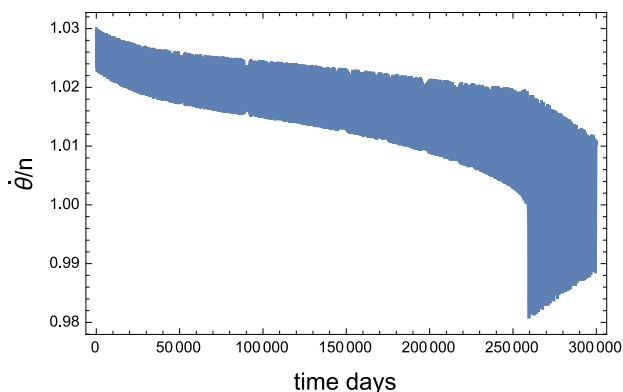


Fig. 5.— Numerically simulated transition of Titan’s spin rate from pseudosynchronous equilibrium to the 1:1 (synchronous) resonance caused by a gradual decline of orbital eccentricity from 0.06 at $t = 0$ to 0.042 at $t = 300000$ days.

delineating the boundaries of our knowledge within the existing analytical models.

Obviously, the second condition of pseudosynchronism (Eq. 22) is automatically satisfied. At first glance, the first condition also appears defunct, because any finite separation of the point of equilibrium secular torque from exact resonance should guarantee capture into this equilibrium in the absence of oscillations. Indeed, there is no triaxial torque, thus, there is no free libration. A more careful analysis, supported by numerical experiments, shows that other, much smaller librations are still present. These are always harmonics of the orbital frequency caused by the periodic terms of the tidal torque, Eq. 5. We observe that the mixed terms $q \neq j$ give both sine- and cosine-harmonics of perturbing polar torque, and hence, similar variations of the spin rate and libration angle. For a given pair q, j , the difference between these harmonics is in the *kvalitet* functions K_c and K_s . Recalling that $G_{20(-1)} = -1/2 e + O(e^3)$, $G_{200} = 1 - 5/2 e^2 + O(e^4)$, $G_{201} = 7/2 e + O(e^3)$, and all other Kaula functions are $O(e^2)$, or higher orders in e , our analysis can be safely limited to the combinations of 0 with -1 or 1 in the (q, j) indices, for a small eccentricity. Furthermore, as long as the spin rate is close to the 1:1 resonance, the largest by magnitude terms among the cos-harmonics will be the ones with $q = \pm 1$ and $j = 0$, because the $q = 0$ term has $K_c(2, 0) = 0$. The *kvalitet* for these two terms is the same, equal to $K_c(2, n)$ (Eq. 11). For the case $M_2 \ll M_1$, this leads to an approximate equation for the combined cos-harmonic of tidal forced libration

$$\ddot{\theta}_{\cos} \approx 6 \frac{M_1}{\xi M_2} n^2 \left(\frac{R}{a} \right)^3 K_c(2, n) e \cos \mathcal{M}. \quad (26)$$

The corresponding forced libration in angular velocity is $\dot{\theta} - n \propto \sin \mathcal{M}$, and in angle $\theta - \mathcal{M} \propto -\cos \mathcal{M}$. Apart from being numerically very small, this perturbation can not violate the pseudosynchronous equilibrium, because it is out of phase with the orbital acceleration of the perturber. Indeed, in the absence of a triaxial force, any restorative force should still be proportional to $+\sin \mathcal{M}$ to lock the body into synchronous rotation.

The sin-harmonics of oscillating tidal torque emerge in the general Darwin-Kaula expansion (Efroimsky 2012a), but, to my knowledge, have not been exploited in the literature. They are proportional to the *kvalitet* functions $K_s(2, \nu)$ (Section 2, Eq. 5), which are, explicitly,

$$K_s(2, \nu) = \frac{3}{2} \frac{\mathcal{A}_2 \nu \tau_M \Re(\nu) + \Re(\nu)^2 + \Im(\nu)^2}{(\Re(\nu) + \mathcal{A}_2 \nu \tau_M)^2 + \Im(\nu)^2}. \quad (27)$$

For negligible Andrade creep contribution ($\tau_A = \infty$), this further simplifies to

$$K_s(2, \nu) = \frac{3}{2} \frac{\mathcal{A}_2 \lambda^2 + \lambda^2 + 1}{\lambda^2 (1 + \mathcal{A}_2)^2 + 1}. \quad (28)$$

where $\lambda = \nu \tau_M$ is the tidal wave number. The maximum value of K_s is reached at $\nu = 0$, $K_s(2, 0) = 3/2$, whereas $K_c(2, \nu_{2200}) \simeq 3/2 \mathcal{A}_2 \nu_{2200} \tau_M$ for $\mathcal{A}_2 \nu_{2200} \tau_M \ll 1$. This indicates that the sin-harmonics may in fact dominate the spectrum of tidal libration. There is, however, another circumstance that may prevent this. For very low effective viscosities and sufficiently

wide orbits, when $n \tau_M \ll 1$, the kvalitet is a slowly declining function of frequency,

$$K_s(2, \nu) = \frac{3}{2} (1 - \mathcal{A}_2(1 + \mathcal{A}_2)\lambda^2 + O(\lambda^4)), \quad (29)$$

so that $K_s(2, n) \approx K_s(2, 0)$. In this case, the pairs of symmetric terms $q, j = 0, 1$ and $1, 0$, and $0, -1$ and $-1, 0$, almost cancel each other, being opposite in sign, and differing by the factor $K_s(2, \nu)$ in magnitude. This happens to our model Titan with a nominal $\tau_M = 0.1$ d considered in the previous simulations. For the sake of completeness, we should consider another specific scenario when $\mathcal{A}_2 n \tau_M \gg 1$. Then, $K_s(2, n) \approx 3/2(1 + \mathcal{A}_2)$ and, limiting the expansion in e to the first power,

$$\ddot{\theta}_{\sin} \approx 6 \frac{M_1}{\xi M_2} n^2 \left(\frac{R}{a} \right)^3 \frac{\mathcal{A}_2}{1 + \mathcal{A}_2} e \sin \mathcal{M}. \quad (30)$$

Due to the positive sign of this periodic force, it acts in a similar fashion to the triaxial torque as a restorative force. A stable pseudosynchronous equilibrium is not possible if the offset of the equilibrium torque in frequency is smaller than the amplitude of the forced libration caused by this periodic tidal torque, which leads to

$$e_{\text{crit}} = \frac{M_1}{\xi M_2} \left(\frac{R}{a} \right)^3 \frac{\mathcal{A}_2}{1 + \mathcal{A}_2}. \quad (31)$$

and the only condition of pseudosynchronism is $e > e_{\text{crit}}$. For Titan, the calculated value of e_{crit} is a rather low $9 \cdot 10^{-5}$. Such small numbers are expected for practically all known star-planet and planet-moon systems because $R \ll a$. In conclusion, we identified one area of parameters for rotationally symmetric bodies ($\mathcal{A}_2 n \tau_M \gg 1$) where pseudosynchronous equilibrium can be precluded by the oscillating components of the tidal torque, but the corresponding threshold eccentricity is so low that such cases should hardly exist.

8. Summary

The rate of energy dissipated in a uniform deformable body by tides in the dominating degree $l = 2$ is

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle &= \frac{\mathcal{G} M_1^2 R^5}{a^6} \sum_{m=0}^2 \frac{(2-m)!}{(2+m)!} (2 - \delta_{0m}) \\ &\times \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} [F_{2mp}(i) G_{2pq}(e)]^2 \nu_{2mpq} K_c(\nu_{2mpq}) \end{aligned} \quad (32)$$

where i is the equator obliquity relative to the orbit plane, $F_{2mp}(i)$ are the inclination functions, and other parameters are defined elsewhere (Efroimsky & Makarov 2014). Due to the presence of the positively definite tidal frequency ν_{2mpq} , the principal resonant mode of the tidal torque is nullified and does not participate in the dissipation of kinetic energy. For example, the main component of the torque for small and moderate eccentricities is the $mpq = 200$ term, but it

vanishes in Eq. 32 if the planet is captured into the 1:1 spin-orbit resonance. The other tidal modes, separated from the dominant mode by an integer times n , will continue to contribute to dissipation. A semiliquid body caught in pseudosynchronous rotation, on the other hand, continues to rotate with respect to the perturber’s vector in the prograde sense, so that the dominant mode of tidal torque continues to generate some dissipation (limited due to the smallness of the tidal frequency ν_{2200}); however, the negative torque from this mode will be exactly balanced, on average, by the component torques from the other modes. As a result, the rate of tidal dissipation is *smaller* at pseudosynchronism than at synchronism. The fact that the heating rate is minimal at the point of tidal equilibrium has been known and discussed in the literature (Wisdom 2008; Levrard 2008).

In this paper, only the “principal” pseudosynchronous state has been considered explicitly, i.e. the one associated with the 1:1 spin-orbit resonance. This limits the analysis to the case of small and moderate eccentricities. The same mathematics can be easily generalized for larger eccentricities ($e \gtrsim 0.2$), where equilibrium torque emerges at other tidal modes. Using the terminology by (Storch & Lai 2014), only the true, or resonant, equilibria will be stable, whereas unstable equilibria of null secular torque can be found between the resonances. These have no practical significance, and they are not considered in this paper. This may be true only for purely Maxwell-type viscoelastic models without any secondary inelastic mechanisms of dissipation, however. The action of an Andrade-type dislocation and unpinning mechanism, for example, coupled with a characteristic low-frequency cutoff (Karato & Spetzler 1990), may conceivably lead to locally stable equilibria between the spin-orbit resonances. Observations of the Lunar tidal phase lag with GRAIL indicate the existence of a secondary peak in the general kvalitet factor k_2/Q between the frequencies $0.075n$ and $1n$ (Williams et al. 2015a). This secondary peak can be heuristically fitted with a Debye peak model (Williams et al. 2015b,c), but the Andrade creep model with a $\tau_A \simeq 0.45$ yr can also reproduce this feature. If the Moon’s eccentricity were higher, the secondary peak of the main resonance could be shifted toward positive values of secular torque by the left shoulder of the 3:2 resonance in such a way that a zero-torque crossing with a locally negative slope takes place between these two resonances. This should make non-resonant rotational equilibria possible in principle. A more detailed analysis supported by numerical simulations is required to investigate this curious possibility.

We have demonstrated that pseudosynchronous rotation is only possible under specific, restrictive conditions. As long as the assumption of a homogeneous body is applicable, it can only happen to rather inviscid, semiliquid or semi-molten bodies with a characteristic tidal wave number $\lambda = \tau_M n \ll 1$. If this condition is not satisfied, the offset of equilibrium secular torque collapses and complete synchronization is the only stable state. Secondly, the orbital eccentricity should be large enough to accommodate the range of free libration to avoid overshooting the resonance at the minimum velocity, per Eq. 22. The threshold value depends on the degree of elongation $(B - A)/C$, hence, more rotationally symmetric bodies have better chances to be pseudosynchronous than their more deformed counterparts. On the other hand, eccentricity may be limited by the first condition for fixed viscosity and shear modulus, see.

Eq. 24. The combination of these opposing criteria implies that pseudosynchronous equilibria are practically impossible for large satellites like Titan if the effective viscosity is greater than $\approx 10^{15}$ Pa.s.

One possibly observable aspect of a stable pseudosynchronous state is that the large-amplitude, long-period free librations of angular velocity around the equilibrium value $\dot{\gamma}_{\text{pseudo}}$ are eternal, i.e., not diminishing in amplitude with time (cf. Fig. 3, right). Physically, this fact is readily explained by the null work produced by the secular tidal torque in that state. For the free libration to be damped, the work of secular torque needs to be negative. This happens when the periodic torque and velocity have opposite phase, whereas in our case, the torque changes its sign over one period of libration but the velocity $\dot{\gamma}$ remains positive. On the other hand, the tidal bulge constantly moves across the surface, the body is being deformed and stressed, and the energy dissipation, being minimized at pseudosynchronism, still goes on. Obviously, the energy spent on tidal heating is not taken from the kinetic energy of the rotating body, but directly from the orbital energy. Inevitably, the orbit should gradually shrink, unless it is supported by a fast rotating perturber.

Being only marginally stable, a pseudosynchronous body can spontaneously leave this state and fall into the 1:1 resonance if the eccentricity falls below the critical value. The tidally driven rate of eccentricity is negative in the vicinity of the 1:1 resonance for small e . Most tidally interacting planets and satellites should therefore be eventually captured in that resonance, unless the eccentricity is boosted by the tides in the primary or other bodies. This may be the reason we do not find any pseudosynchronously rotating satellites in the Solar system. Titan, with its current value of eccentricity, could not be pseudosynchronous, but if its orbit was more elongated in the past, it could be pseudosynchronous for a limited duration.

Acknowledgments

I thank the anonymous referee for the exceptionally thorough and constructive review of the paper. Generous help and support from M. Efroimsky, B. Noyelles, W. Henning, and J. Frouard are also greatly appreciated.

REFERENCES

- Auclair-Desrotour, P., Le Poncin-Lafitte, C., & Mathis, S. 2014, A&A, 561L, 7
- Bills, B.G. , & Nimmo, F. 2008, Icarus, 196, 293
- Correia, A.C.M., et al. 2014, A&A, 571, A50
- Danby, J.M.A. 1962, Fundamentals of celestial mechanics, MacMillan, New York
- Doremus R.H. 2002, *Journal of Applied Physics*, 92, 7619

- Efroimsky, M. 2012, CeMDA, 104, 257
- Efroimsky, M. 2012, ApJ, 746, 150
- Efroimsky, M., & Williams, J.G. 2009, *Celestial Mechanics and Dynamical Astronomy*, 104, 257
- Efroimsky M., & Makarov V.V., 2014, ApJ, 795, 6
- Efroimsky, M. & Lainey, V. 2007, J. of Geophys. Research, 112, E12003
- Ferraz-Mello S., Rodrigues A., Hussmann, H. 2008, CMDA, 101, 171
- Ferraz-Mello S. 2013, CMDA, 116, 109
- Geissler P.E., et al. 1998, Nature, 391, 368
- Goldreich P. 1966, AJ, 71, 1
- Goldreich P. & Peale S.J. 1968, ARA&A, 6, 287
- Goldreich P. & Peale S.J. 1966, AJ, 71, 425
- Henning, W.G., & Hurford, T. 2014, ApJ, 789, 30
- Hussmann, H., et al. 2010, Space Sci. Rev., 153, 317
- Hut P., 1980, A&A, 92, 167
- Hut P., 1981, A&A, 99, 126
- Karato, S., & Spetzler, H.A. 1990, Rev. of Geophysics, 28, 399
- Kaula W.M. 1964, Rev. Geophys. Space Phys., 2, 661
- Lainey V., et al. 2012, ApJ, 752, 14
- Lambeck K. 1975, J. Geophys. Res., 80, 2917
- Levrard B. 2008, Icarus, 193, 641
- Makarov V.V. 2012, ApJ, 752, 73
- Makarov V.V. 2013, MNRAS, 434, L21
- Makarov V.V., Berghea C., Efroimsky, M. 2014, ApJ, 780, 124
- Makarov V.V., & Efroimsky M., 2013, ApJ, 764, 27
- Mignard F. 1980, Moon Planets, 23, 185

- Mignard F. 1981, *Moon Planets*, 24, 189
- Murray, C.D., & Dermott, S.F. 1999, *Solar System Dynamics*. Cambridge University Press, Cambridge UK
- Noyelles, B., et al. 2014. *Icarus*, 241, 26
- Remus, F., Mathis, S., & Zahn, J.-P. 2012, *A&A*, 544, A132
- Rodríguez, A., Ferraz-Mello, S., Hussmann, H. 2008, In: Sun, Y.S., Ferraz-Mello, S., Zhou, J.L. (eds.) *Exoplanets: Detection, Formation and Dynamics*. Proceedings of the IAU Symposium No249, 179
- Shoji, D., & Kurita, K. 2014, *ApJ*, 789, 3
- Stamenković, V., Noack, L., Breuer, D., & Spohn, T. 2012, *ApJ*, 748, 41
- Storch, N.I., & Lai, D. 2014, *MNRAS*, 438, 1526
- Tiscareno, M., Thomas, P.C., & Burns, J.A. 2009, *Icarus*, 204, 254
- Williams, J.G., & Efroimsky, M. 2012, *CMDA*, 114, 387
- Williams, J.G., et al. 2015, *LPI*, 46, 1380
- Williams, J.G., Boggs, D.H., Ratcliff, J.T. 2015, *LPI*, 46, 1877
- Williams, J.G., et al. 2015, *Icarus*, in press
- Wisdom J. 2008, *Icarus*, 193, 637
- Zahn J.-P. 1977, *A&A*, 57, 383